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A MODEL OF THE STRUCTURE OF HOMOGENEOUS TURBULENCE.(U)
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6 A MODEL OF THE STRUCTURE OF
HOMOGENEOUS TURBULENCE

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I. INTRODUCTION

Following References 1 and 2, we propose to generalize well-known transport equations for turbulent flows so that they are assumed to apply to two-point correlation functions. We use analogy with their one-point counterparts to which they are required to reduce as the separation of the two points is made to vanish.

In order to illustrate the development of this program, we concentrate in this report on homogeneous turbulence at high Reynolds numbers so that the two one-point models needed are

the dissipation model

$$-2\nu\left\langle\frac{\partial u_i'}{\partial x_\alpha}\frac{\partial u_j'}{\partial x_\alpha}\right\rangle = -\left(2b\,\frac{q}{\Lambda}\right)\left(\frac{q^2}{3}\right)\delta_{ij} \qquad , \left(b \approx \frac{1}{8}\right) \tag{1}$$

and the redistribution model,

$$\left\langle p'\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)\right\rangle = -\left(c\,\frac{q}{\Lambda}\right)\left[\overline{u_i'u_j'} - \frac{q^2}{3}\,\delta_{ij}\right] \quad , (c \approx 1) \tag{2}$$

Both models have been used successfully for such a long time that it is difficult to assign rigidly their originators. They have acquired special importance in the recent developments of turbulent transport calculations. (Ref. 3). These models have rather obvious generalizations to two-point correlations (they will be given in the next section). If such generalizations are made, then the rate equation for $R_{ij} = \langle u_i'(x)u_j'(y)\rangle$ can be closed and, in principle, solved, with the result that the structure as well as the scale of the turbulent field can be determined.

A. For homogeneous flows, using the leading term in the moment expansion of R_{ij}

$$R_{ij} = \frac{q^2}{3}\,\Lambda_{ij}\,\delta(|r|)$$

to obtain a closed scale equation is permissible because the correlation length is infinitely small compared with the size of the system.

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1. Sandri, G., "A New Approach to the Development of Scale Equations for Turbulent Flows," A.R.A.P. Report No. 302, April 1977.
2. Sandri, G., "Recent Results Obtained in the Modeling of Turbulent Flows by Second-Order Closure," AFOSR-TR-78-0680, February 1978.
3. Donaldson, C.duP., "Construction of a Dynamic Model of the Production of Atmospheric Turbulence and the Dispersal of Atmospheric Pollutants," *Workshop on Micrometeorology* (D.A.Haugen, ed.), American Meteorological Society, Boston (1973), pp. 313-392.

B. Experimental information in available on several variables for both isotropic grid turbulence and homogeneous shear turbulence (Refs. 4 and 5).

With only two adjustable constants, the model covers qualitatively both types of turbulent flows. We shall show this by exhibiting explicit analytic solutions with several of the desired features. In particular, the analytic solutions for the homogeneous turbulence models show the presence of two distinct time scales which characterize, respectively, the rapid settling of the tensor character of the flow to an asymptotic state and the slower development of the energy and mean scale. It is found that for both grid and homogeneous shear turbulence, the ratio of the two scales is about ten.

The fast time is the redistribution time, Λ/q , while the slower one is $b\Lambda/q$ (dissipation scale) for grid turbulence and $v'\Lambda/q$ (merging scale) for shear flows (v' is defined in the next section) after an initial transient. This feature of the model solutions seems to be well reflected in the data.

In this report, we will construct the general structure equations (Appendix A). We shall also obtain a first-order test of our model by exploiting the following result for homogeneous turbulence:

> The equation for R_{ij} is wholly determined from three requirements:
> (i) that it should yield the observed transport for $\overline{u_i u_j}$;
> (ii) that it should satisfy the continuity equation;
> (iii) that it should yield the correct limit for isotropic turbulence.

We now consider briefly an analogy between the models (1) and (2) and the Newton-Fourier heat equation. We may think of the Newton-Fourier equation,

$$q_i = -K \frac{\partial T}{\partial x_i} \tag{3}$$

which gives the heat flow vector q_i in terms of the temperature gradient, as a phenomenological law (or, more incisively, "model") which allows us to close the heat equation and hence gives us a chance to solve it. As a model, (3) is subject to restrictions. It is, however, tensorial and therefore independent of geometry, hence valid in any coordinate system. Fourier luckily relied (intuitively) on the tensor nature of (3) and overlooked the restrictions: (i) K is temperature-dependent even for the simplest material (inert gases); (ii) K is a tensor for any nonsimple material; (iii) in the presence of electricity, q_i requires altogether a new term (the thermoelectric effect of Thomson).

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4. Harris, V.G., Graham, J.A.H. and Corrsin, S., "Further Experiments in Nearly Homogeneous Turbulent Shear Flow," J. Fluid Mech. 81, 1977, pp. 657-687. Corrigendum, J. Fluid Mech. 86, 1978, pp. 795-796.

5. Corrsin, S. and Kollman, W., "Preliminary Report on Suddenly Sheared Cellular Motion as a Qualitative Model of Homogeneous Turbulent Shear Flow," Proc. SQUID Symp. on Turbulence in Internal Flows, pp. 11-33 (S.N.B.Murthy, ed.), Hemisphere Publishing Co., 1977.

Of course, analogous remarks may well apply to (1) and (2). It seems to us that Fourier's work encourages the view that a good treatment of the simplest model is desirable.

II. DERIVATION OF THE CLOSED EQUATIONS FOR THE REYNOLDS AND SCALE TENSORS

We start with the Navier-Stokes equations for the velocity field u_i and the kinematic pressure p (pressure/density)

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i \tag{4}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{5}$$

The velocity and pressure are decomposed, following Reynolds, into mean and fluctuating parts

$$u_i = \overline{u_i} + u_i' \tag{6}$$

$$p = \bar{p} + p' \tag{7}$$

Substituting (6) and (7) into (4) and (5), one obtains separate equations for the mean and fluctuations. After standard operations, these equations can be cast into the form of equations for the correlation tensor R_{ij}

$$R_{ij}(x,y) = \left\langle u_i'(x)u_j'(y)\right\rangle \tag{8}$$

namely,

$$\begin{aligned}\frac{\partial R_{ij}}{\partial t} = &- \left[\bar{u}_k(x) \frac{\partial}{\partial x_k} + \bar{u}_k(y) \frac{\partial}{\partial y_k}\right] R_{ij} + \\ &- \left[R_{ik} \frac{\partial \bar{u}_j}{\partial y_k} + \frac{\partial \bar{u}_i}{\partial x_k} R_{kj}\right] + \\ &- \left[\frac{\partial}{\partial x_k} \left\langle u_i'(x)u_k'(x)u_j'(y)\right\rangle + \frac{\partial}{\partial y_k}\left\langle u_i'(x)u_k'(y)u_j'(y)\right\rangle\right] + \\ &- \left[\left\langle \frac{\partial p'}{\partial x_i} u_j'(y)\right\rangle + \left\langle u_i'(x) \frac{\partial p'}{\partial y_j}\right\rangle\right] + \nu\left[\left(\nabla_x^2 + \nabla_y^2\right)R_{ij}\right]\end{aligned} \tag{9}$$

and

$$\frac{\partial R_{ij}}{\partial x_i} = 0 \tag{10}$$

$$\frac{\partial R_{ij}}{\partial y_j} = 0 \tag{11}$$

It is convenient to recast these equations by introducing "centroid" and "relative" variables as follows:

$$x_c = \frac{x + y}{2} \tag{12}$$

$$r = y - x \tag{13}$$

Using the chain rule for differentiation, the equations for R_{ij} become

$$\begin{aligned}
\frac{\partial R_{ij}}{\partial t} = & -\left[\frac{u_k(x) + u_k(y)}{2} \frac{\partial}{\partial x_{ck}} + \left(u_k(y) - u_k(x)\right) \frac{\partial}{\partial r_k} \right] R_{ij} + \\
& - \left[R_{ik} \frac{\partial \bar{u}_j}{\partial x_{ck}} + \frac{\partial \bar{u}_i}{\partial x_{ck}} R_{kj} \right] + \\
& - \frac{1}{2} \frac{\partial}{\partial x_{ck}} \left[\left\langle u_i'(x) u_k'(x) u_j'(y) \right\rangle + \left\langle u_i'(x) u_k'(y) u_j'(y) \right\rangle \right] + \\
& + \frac{\partial}{\partial r_k} \left[\left\langle u_i'(x) u_k'(x) u_j'(y) \right\rangle - \left\langle u_i'(x) u_k'(y) u_j'(y) \right\rangle \right] + \\
& - \left[\frac{\partial}{\partial x_{ci}} \left\langle p'(x) u_j'(y) \right\rangle + \frac{\partial}{\partial x_{cj}} \left\langle u_i'(x) p'(y) \right\rangle \right] + \\
& + \left[\left\langle p'(x) \frac{\partial u_j'(y)}{\partial x_{ci}} \right\rangle + \left\langle \frac{\partial u_i'(x)}{\partial x_{cj}} p'(y) \right\rangle \right] + \\
& + \nu \nabla_c^2 R_{ij} - 2\nu \left\langle \frac{\partial u_i'(x)}{\partial x_{ck}} \frac{\partial u_j'(y)}{\partial x_{ck}} \right\rangle
\end{aligned} \tag{14}$$

and

$$\frac{\partial R_{ij}}{\partial x_{ci}} = 2\,\frac{\partial R_{ij}}{\partial r_i} \tag{15}$$

$$\frac{\partial R_{ij}}{\partial x_{cj}} = -2\,\frac{\partial R_{ij}}{\partial r_j} \tag{16}$$

where the arguments in the correlations are given by

$$x = x_c - \frac{r}{2} \tag{17}$$

$$y = x_c + \frac{r}{2} \tag{18}$$

For homogeneous turbulence, the derivatives with respect to the centroid vanish when acting on any correlation (but not, in general, on the mean velocity) or on a fluctuation. We then have, for homogeneous turbulence,

$$\begin{aligned}\frac{\partial R_{ij}}{\partial t} = &-\left[\bar{u}_k(y) - \bar{u}_k(x)\right]\frac{\partial R_{ij}}{\partial r_k} - \left[R_{ik}\,\frac{\partial \bar{u}_j}{\partial x_{ck}} + \frac{\partial \bar{u}_i}{\partial x_{ck}}\,R_{kj}\right] + \\ &+ \frac{\partial}{\partial r_k}\left[\left\langle u_i'(x)u_k'(x)u_j'(y)\right\rangle - \left\langle u_i'(x)u_k'(y)u_j'(y)\right\rangle\right] + \\ &+\left[\left\langle p'(x)\,\frac{\partial u_j'(y)}{\partial x_{ci}}\right\rangle + \left\langle \frac{\partial u_i'(x)}{\partial x_{cj}}\,p'(y)\right\rangle\right] + \\ &- 2\nu\left\langle \frac{\partial u_i'(x)}{\partial x_{ck}}\,\frac{\partial u_j'(y)}{\partial x_{ck}}\right\rangle\end{aligned} \tag{19}$$

and

$$\frac{\partial R_{ij}}{\partial r_i} = 0 \tag{20}$$

$$\frac{\partial R_{ij}}{\partial r_j} = 0 \tag{21}$$

The terms in Eq. (19) that prevent closure of the equation are the last three. In order to achieve closure, we introduce generalized transport models as follows:

Dissipation equation

$$-2\nu \left\langle \frac{\partial u_i'(x)}{\partial x_{ck}} \frac{\partial u_j'(y)}{\partial x_{ck}} \right\rangle = -2\left(\frac{bq}{\Lambda}\right)\frac{\delta_{ij}}{3} R_{\alpha\alpha}(r) \tag{22}$$

Intercomponent rearrangement

$$\left\langle p'(x) \frac{\partial u_j'(y)}{\partial x_{ci}} + \frac{\partial u_i'(x)}{\partial x_{cj}} p'(y) \right\rangle = -\frac{q}{\Lambda}\left[R_{ij} - \frac{1}{3}\delta_{ij}(R_{\alpha\alpha} + T_{ij})\right] \tag{23}$$

where T_{ij} is assumed to satisfy

$$T_{ij}(r = 0) = 0 \tag{24}$$

$$\int \frac{T_{ij}(r)}{4\pi r^2} d^3r = 0 \tag{25}$$

These properties are utilized in Section III to derive the Reynolds stress and scale tensor equations. For a full determination of eddy structure, additional properties are needed which will be discussed in Appendix A. In addition to the two generalizations given above, we need a model for the spatially homogeneous part of the triple velocity correlation which represents the nonlinear effects of local turbulent convection. These nonlinear effects correspond to either eddy break-up (cascading, when wave vectors add) or merging of eddies (when the wave vectors subtract). We thus call the model

Eddy size rearrangement

$$\frac{\partial}{\partial r_k}\left[\left\langle u_i'(x)u_k'(x)u_j'(y)\right\rangle - \left\langle u_i'(x)u_k'(y)u_j'(y)\right\rangle\right] = v' \frac{q}{\Lambda}\left[R_{ij} - N_{ij}\right] \tag{26}$$

where N_{ij} satisfies

$$N_{ij}(r = 0) = \overline{u_i'u_j'} \tag{27}$$

$$\int \frac{N_{ij}}{4\pi r^2} d^3r = 0 \tag{28}$$

As in the case of the tensor K_{ij} for a full specification of structure, N_{ij} must satisfy additional requirements (see Section III). Simple examples of N_{ij} are

$$\frac{\partial^n}{\partial |r|^n}\left(|r|^n R_{ij}\right) \quad , n \text{ is a positive integer}$$

Substituting the models (22), (23) and (26) into (19) yields

$$\frac{\partial R_{ij}}{\partial t} = -\left[\bar{u}_k(y) - \bar{u}_k(x)\right]\frac{\partial R_{ij}}{\partial r_k} - \left[R_{ik}\frac{\partial \bar{u}_j}{\partial x_{ck}} + \frac{\partial \bar{u}_i}{\partial x_{ck}} R_{kj}\right] +$$

$$+ v' \frac{q}{\Lambda}\left[R_{ij} - N_{ij}\right] - \frac{q}{\Lambda}\left[R_{ij} - \frac{1}{3}\delta_{ij}(R_{\alpha\alpha} + T_{ij})\right] +$$

$$- 2\left(\frac{bq}{\Lambda}\right)\frac{\delta_{ij}}{3} R_{\alpha\alpha} \qquad (29)$$

with (20) and (21), i.e., continuity, holding.

To obtain the rate equation for the Reynolds stress (divided by ρ), we let $r \to 0$ in (29) and find, using (24) and (27)

$$\frac{\partial}{\partial t}\overline{u_i' u_j'} = -\left[\overline{u_i' u_k'}\frac{\partial \bar{u}_j}{\partial x_{ck}} + \frac{\partial \bar{u}_i}{\partial x_{ck}}\overline{u_k' u_j'}\right] - \frac{q}{\Lambda}\left[\overline{u_i' u_j'} - \frac{1}{3}\delta_{ij}q^2\right]$$

$$- 2\left(\frac{bq}{\Lambda}\right)\frac{1}{3}\delta_{ij}q^2 \qquad (30)$$

To obtain a rate equation for the scale tensor Λ_{ij},

$$\frac{q^2}{3}\Lambda_{ij} \equiv \int \frac{R_{ij}}{4\pi r^2}\, d^3 r \qquad (31)$$

that includes the first order information on turbulence structure, we expand $R_{ij}(r)$ in terms of its moments and retain the lowest terms. We obtain

$$R_{ij}(r) = \frac{q^2}{3}\Lambda_{ij}\delta(|r|) \qquad (32)$$

where the Dirac function of the magnitude of r is related to $\delta(r)$ by

$$\delta(r) = \frac{\delta(|r|)}{4\pi r^2} \tag{33}$$

The approximation (33), as we remarked in the Introduction, is best justified for homogeneous turbulence because in this case the spatial scale of the mean flow is infinite.

We now apply to (29) the operation $\int d^3r/4\pi r^2$ and, using (25) and (28), we find

$$\frac{\partial}{\partial t}\left(q^2\Lambda_{ij}\right) = -q^2\left[\Lambda_{ik}\frac{\partial\bar{u}_j}{\partial x_{ck}} + \frac{\partial\bar{u}_i}{\partial x_{ck}}\Lambda_{kj}\right] + \left(v'\,\frac{q}{\Lambda}\right)q^2\Lambda_{ij} +$$

$$- \frac{q}{\Lambda}\left[q^2\Lambda_{ij} - \frac{1}{3}\delta_{ij}q^2\Lambda_{kk}\right] - 2bq^3\delta_{ij} \tag{34}$$

Convection of Λ_{ij} does not occur for homogeneous turbulence. To see that this fact is a consequence of the moment approximation (32), we note that (33) gives

$$\int\frac{d^3r}{4\pi r^2}\left[\bar{u}_k\left(x_c + \frac{r}{2}\right) - \bar{u}_k\left(x_c - \frac{r}{2}\right)\right]\frac{\partial}{\partial r_k}\,\delta(|r|)$$

$$= \int\frac{d^3r}{4\pi r^2}\left[\bar{u}_k\left(x_c + \frac{r}{2}\right) - \bar{u}_k\left(x_c - \frac{r}{2}\right)\right]\frac{\partial}{\partial r_k}\left(4\pi r^2\delta(r)\right)$$

$$= - \int d^3r\;\delta(r)\left[\frac{\partial}{\partial x_{ck}}\,\bar{u}_k\left(x_c + \frac{r}{2}\right) + \frac{\partial}{\partial x_{ck}}\,\overline{u_k}\left(x_c - \frac{r}{2}\right)\right]$$

$$= - 2\,\frac{\partial\bar{u}_k(x_c)}{\partial x_{ck}} = 0$$

where we performed an integration by parts and used continuity. We obtain the final form of the equation for the scale tensor by substituting into (34)

$$\frac{\partial}{\partial t}\left(q^2\Lambda_{ij}\right) = q^2\,\frac{\partial\Lambda_{ij}}{\partial t} + \Lambda_{ij}\,\frac{\partial q^2}{\partial t} \tag{35}$$

$$\frac{\partial q^2}{\partial t} = - 2\,\overline{u_i'u_k'}\,\frac{\partial \overline{u_k}}{\partial x_{ci}} - 2b\,\frac{q^3}{\Lambda} \tag{36}$$

This latter, the energy equation, is obtained by contracting (30). The result is

$$\frac{\partial \Lambda_{ij}}{\partial t} = -\left[\Lambda_{ik}\frac{\partial \bar{u}_j}{\partial x_{ck}} + \frac{\partial \bar{u}_i}{\partial x_{ck}}\Lambda_{kj}\right] + \Lambda_{ij}\left[2\,\frac{\overline{u_k'u_\ell'}}{q^2}\frac{\partial \bar{u}_k}{\partial x_{c\ell}} + \frac{q}{\Lambda}\,(2b + v')\right] +$$

$$- \frac{q}{\Lambda}\left[\Lambda_{ij} - \frac{\delta_{ij}}{3}\Lambda_{kk}\right] - 2bq\delta_{ij} \tag{37}$$

Note that Eq. (37) is obtained by dividing by q^2 and, therefore, it should not be used in the absence of turbulence.

To close the pair (30) and (37), we choose

$$\Lambda = \frac{1}{3}\Lambda_{kk} \tag{38}$$

The reason for this choice was discussed in a previous report (Ref. 1). Contraction of (37) and use of (38) yields the equation for the mean scale

$$\frac{\partial \Lambda}{\partial t} = - \frac{2}{3}\Lambda_{ik}\frac{\partial \bar{u}_i}{\partial x_{ck}} + \Lambda\,\frac{2\overline{u_k'u_\ell'}}{q^2}\frac{\partial \bar{u}_k}{\partial x_{c\ell}} + v'q \tag{39}$$

It is interesting to note that, in this model, the coefficient of the "production" term is not a universal constant and receives generally competing contributions from the Reynolds stress and from the tensor scale.

III. SOLUTIONS OF THE COUPLED EQUATIONS FOR STRESS AND SCALE TENSOR

In this section we give two analytic solutions to the coupled equations for the stress tensor and scale tensor equations. We use subsections to separate the different calculations.

A. Equations in Standard Coordinates

The centroid vector and mean velocity vector are taken to have components

$$(x,y,z) \ , \ (V(y),0,0)$$

with $\partial U/\partial y = U' = \text{constant}$. The relevant components of the Reynolds stress equations are obtained from Eqs. (30) and (36). We drop primes on the fluctuations and give a form useful for numerical integration in which $\overline{u_1^2}$ and Λ_{11} are calculated from

$$\overline{u_1^2} = q^2 - \overline{u_2^2} - \overline{u_3^2} \tag{40}$$

$$\Lambda_{11} = 3\Lambda - \Lambda_{22} - \Lambda_{33} \tag{41}$$

The other relevant components of the stress and energy equations are

$$\frac{\partial}{\partial t}\overline{u_2^2} = \frac{1}{3}(1 - 2b)\frac{q^3}{\Lambda} - \frac{q}{\Lambda}\overline{u_2^2} \tag{42}$$

$$\frac{\partial}{\partial t}\overline{u_3^2} = \frac{1}{3}(1 - 2b)\frac{q^2}{\Lambda} - \frac{q}{\Lambda}\overline{u_3^2} \tag{43}$$

$$\frac{\partial}{\partial t}\overline{u_1u_2} = -u_2^2 \ U' - \frac{q}{\Lambda}\overline{u_1u_2} \tag{44}$$

$$\frac{\partial q}{\partial t} = -\frac{\overline{u_1u_2}}{q}U' - b\frac{q^2}{\Lambda} \tag{45}$$

For the tensor scale components, we obtain, using Eqs. (37) and (39)

$$\frac{\partial}{\partial t}\Lambda_{22} = -\frac{1}{T}\Lambda_{22} + (1 - 2b)q \tag{46}$$

$$\frac{\partial}{\partial t}\Lambda_{33} = -\frac{1}{T}\Lambda_{33} + (1 - 2b)q \tag{47}$$

$$\frac{\partial}{\partial t}\Lambda_{12} = -\frac{1}{T}\Lambda_{12} - U'\Lambda_{22} \tag{48}$$

$$\frac{\partial}{\partial t}\Lambda = 2\,\frac{\overline{u_1u_2}}{q^2}\,U'\Lambda - \frac{2}{3}\Lambda_{12}U' + v'q \tag{49}$$

where

$$\frac{1}{T} = -2\,\frac{\overline{u_1u_2}}{q^2}\,U' + \frac{q}{\Lambda}(1 - 2b - v') \tag{50}$$

B. *Solution of the Shearless Equations*

Setting $U' = 0$, we see that equations for q and Λ decouple from the tensor components. Introducing the deviators

$$d_{ij} = \overline{u_iu_j} - \frac{1}{3}\delta_{ij}q^2 \tag{51}$$

$$D_{ij} = \Lambda_{ij} - \frac{1}{3}\delta_{ij}\Lambda_{kk} \tag{52}$$

and the time

$$\tau = \frac{\Lambda}{q} \tag{53}$$

We have the set

$$\frac{\partial}{\partial t}q = -\frac{b}{\tau}q \quad , \quad \frac{\partial}{\partial t}\Lambda = \frac{v'}{\tau}\Lambda \tag{54}$$

$$\frac{\partial}{\partial t}d_{ij} = -\frac{b}{\tau}d_{ij} \,, \quad \frac{\partial}{\partial t}D_{ij} = -\frac{1 - 2b - v'}{\tau}D_{ij} \tag{55}$$

$$\frac{\partial}{\partial t}\tau = b + v' \tag{56}$$

Integrating (56) as

$$\tau = (b + v')(t - t_o) + \frac{\Lambda_o}{q_o} \tag{57}$$

we see that q , Λ , d_{ij} and D_{ij} are suitable powers of $(q_o/T_o)\tau$; for example,

$$q = q_o\left[(b + v') \frac{q_o}{\Lambda_o} (t - t_o) + 1\right]^{-b/(b+v')} \tag{58}$$

$$d_{ij} = d_{ij_o}\left[(b + v') \frac{q_o}{\Lambda_o} (t - t_o) + 1\right]^{-1/(b+v')} \tag{59}$$

$$\Lambda = \Lambda_o\left[(b + v') \frac{q_o}{\Lambda_o} (t - t_o) + 1\right]^{-v'/(b+v')} \tag{60}$$

From grid turbulence data on q and Λ , we may choose

$$b \approx \frac{1}{8} \ , \quad v' \approx 0.075 \tag{61}$$

We then see that for large times

$$q \sim q_o \left[\cdot\ 2 \frac{q_o}{\Lambda_o} t \right]^{-5/8} \tag{62}$$

$$d_{ij} \sim d_{ij_o}\left[\cdot\ 2 \frac{q_o}{\Lambda_o} t \right]^{-5} \tag{63}$$

which shows that the deviator decays with a power about four times larger than the energy. This substantial difference may eventually be checked in our anisotropic grid flow.

From the solutions given above, we can verify that statistics are preserved by the model equations if the model parameters satisfy certain bounds. We first show that the two tensors $\overline{u_i u_j}$ and Λ_{ij} are positive definite from their definitions. Consider an arbitrary (constant) A_i then,

$$A_i\overline{u_i u_j}A_j = \overline{(u \cdot A)^2} \geq 0 \tag{64}$$

the equality sign holding for $A \equiv 0$ only. Thus $\overline{u_i u_j}$ is a positive definite tensor.

From the definition of the scale tensor given by Eq. (31), using Fourier transform on R_{ij} ,

$$\frac{q^2}{3}\Lambda_{ij} = \int \frac{R_{ij}}{4\pi r^2}\, d^3r = \int \frac{\phi_{ij}}{8\pi k}\, d^3k \tag{65}$$

where the power spectrum tensor ϕ_{ij} is positive definite by Khiutchine's theorem. Thus,

$$\frac{q^2}{3} A_i\Lambda_{ij}A_j = \int \frac{d^3k}{8\pi k} A_i\phi_{ij}A_j \geq 0 \tag{66}$$

Thus, Λ_{ij} is positive definite because q^2 is positive as a consequence of (64).

Using the solution (61) and an analogous solution for Λ_{ij}, we find

$$\overline{u_iu_j}(t) = \overline{u_iu_j}(0)\left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')} + \frac{1}{3}\delta_{ij}q_o^2\left[\left(\frac{q_o}{\Lambda_o}\tau\right)^{-2b/(b+v')} - \left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')}\right] \tag{67}$$

$$\Lambda_{ij}(t) = \Lambda_{ij}(0)\left(\frac{q_o}{\Lambda_o}\tau\right)^{-(1-2b-v')/(b+v')} + \Lambda(0)\delta_{ij}\left[\left(\frac{q_o}{\Lambda_o}\tau\right)^{v'/(b+v')} - \left(\frac{q_o}{\Lambda_o}\tau\right)^{-(1-2b-v')/(b+v')}\right] \tag{68}$$

We now multiply (67) by A_iA_j when A_i is an arbitrary vector and find

$$A_i\overline{u_iu_j}(t)A_j = \overline{(A\cdot u)^2}(0)\left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')} + \frac{A^2}{3}q_o^2\left[\left(\frac{q_o}{\Lambda_o}\tau\right)^{-2b/(b+v')} - \left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')}\right] \tag{69}$$

From (57) we see that

$$\frac{q_o}{\Lambda_o}\tau \geq 1 \qquad (L + v' \geq 0) \tag{70}$$

Sufficient for the left-hand side of (69) to be positive is

$$\left(\frac{q_o}{\Lambda_o}\tau\right)^{-2b/(b+v')} \geq \left(\frac{q_o}{\Lambda_o}\tau\right)^{-1/(b+v')} \tag{71}$$

which requires, using (70),

$$2b \leq 1 \tag{72}$$

A similar analysis applies to Λ_{ij} ; however, no further restrictions on the parameters is found.

C. An Exact Solution of the Equations with Shear

To obtain a solution of the equations with shear, we let

$$\begin{aligned}
q &= V\, e^{av't} & \Lambda &= L\, e^{av't} \\
\overline{u_1^2} &= W_1\, e^{2av't} & \Lambda_{11} &= L_1\, e^{av't} \\
\overline{u_2^2} &= W_2\, e^{2av't} & \Lambda_{22} &= L_2\, e^{av't} \\
\overline{u_3^2} &= W_3\, e^{2av't} & \Lambda_{33} &= L_3\, e^{av't} \\
\overline{u_1 u_2} &= W_4\, e^{2av't} & \Lambda_{12} &= L_4\, e^{av't}
\end{aligned} \tag{73}$$

Substituting these forms into the differential equations of III.A, we find that the exponentials cancel and that an algebraic set of equations for the amplitudes are obtained. It is possible, with some algebra, to solve the amplitude equations explicitly in terms of the parameters b and v' . The energy components are

$$\frac{\overline{u_1^2}}{q^2} = \frac{1 + 6v' + 4b}{3(1 + 2v')} \tag{74}$$

$$\frac{\overline{u_2^2}}{q^2} = \frac{u_3^2}{q^2} = \frac{1 - 2b}{3}\,\frac{1}{1 + 2v'} \tag{75}$$

The scale components are

$$\frac{\Lambda_{11}}{\Lambda} = \frac{1 + 6v' + 4b}{1 + 2v'} \tag{76}$$

$$\frac{\Lambda_{22}}{\Lambda} = \frac{\Lambda_{33}}{\Lambda} = \frac{1 - 2b}{1 + 2v'} \tag{77}$$

We see that

$$\frac{\Lambda_{11}}{\Lambda_{22}} = \frac{\overline{u_1^2}}{\overline{u_2^2}} = \frac{1 + 6v' + 4b}{1 - 2b} \tag{78}$$

The off-diagonal components are

$$Br = \frac{|\overline{u_1 u_2}|}{q^2} = \frac{1}{1 + 2v'} \sqrt{\frac{(1 - 2b)(b + v')}{3}} \tag{79}$$

$$\frac{\Lambda_{12}}{\Lambda} = - \frac{1}{1 + 2v'} \sqrt{3(1 - 2b)(b + v')} \tag{80}$$

The Corrsin parameter is

$$C \equiv \frac{\overline{u_1 u_2}}{\sqrt{\overline{u_1^2}\,\overline{u_2^2}}} = \sqrt{\frac{3(b + v')}{1 + 4b + 6v'}} \tag{81}$$

The ratio of the two times is

$$\frac{1}{\alpha} = \frac{q}{U'\Lambda} = \frac{1}{1 + 2v'} \sqrt{\frac{1 - 2b}{3(b + v')}} \tag{82}$$

and the growth rate, a, is

$$a = v' \frac{1}{\alpha} = \frac{v'}{1 + 2v'} \sqrt{\frac{1 - 2b}{3(b + v')}} \tag{83}$$

We notice two additional interesting parameters:

$$\frac{|\overline{u_1 u_2}|}{q^2} \frac{U'}{q} = \alpha \cdot Br = b + v' \tag{84}$$

$$\frac{\Lambda_{ij}}{\Lambda} - \delta_{ij} = 3\left(\frac{\overline{u_i u_j}}{q^2} - \frac{1}{3} \delta_{ij}\right) \qquad (\sigma \equiv 3) \tag{85}$$

We make the following remarks:

1) Several numerical integrations suggest that any solution that initially statistically realizable will remain so and will asymptote to the exact solutions given.

2) If the scalar scale equation

$$\frac{\partial \Lambda}{\partial t} = c \frac{\Lambda}{2} \overline{u_1 u_2} \, U' + v'q \qquad (c \approx 0.35) \tag{86}$$

is adopted instead of the tensor equation, an exponential solution exists and has the same qualitative features insofar as scale and energy are concerned (of course, no scale directivity results). This indicates that the models for scale, which were not designed to fit homogeneous shear data, are quite stable.

3) The equations for R_{ij} itself (or the spectrum ϕ_{ij}, see Appendix A) have the same type of convective equilibrium solutions.

IV. CONCLUSIONS

Our analysis of a model of homogeneous turbulence, deduced from simple assumptions made on the two-point tensor R_{ij} , has brought out four main conclusions.

1. The model contains only two adjustable parameters which we chose to fix from grid turbulence data. Good qualitative agreement with homogeneous shear flow results.

2. The model implies the existence of two distinct time scales which are separated by a factor of about 10. They appear after an initial transient phase has died out during which q/Λ becomes approximately equal to U' . On the fast scale, $\Lambda/q \approx (U')^{-1}$, the normalized deviator

$$a_{ij} = \frac{\overline{u_i' u_j'}}{q^2} - \frac{1}{3}\delta_{ij}$$

   locks into a constant value indicated by a Corrsin parameter

$$\frac{|\overline{u_1' u_2'}|}{\sqrt{\overline{u_1'^2}\;\overline{u_2'^2}}} \sim 0.5$$

   or by a Bradshaw number

$$\frac{|\overline{u_1' u_2'}|}{q^2} \sim 0.19$$

   On the slower scale, $v'(\Lambda/q) \sim 0.07(\Lambda/q)$, the energy components and the mean scale grow exponentially. This solution can be thought of as a superequilibrium with convection. A qualitative physical picture is as follows. During the initial transient, when $(q_0/\Lambda_0) \ll U'$, the shear brings the sudden distortion-like turbulence up to convective equilibrium $(q/\Lambda \approx U')$, while when $(q_0/\Lambda_0) \gg U'$, the turbulence decays grid-like to convective equilibrium. Then a merging mechanism takes over (a multi-layer Brown-Roshko effect) so that eddies fold with each other, making larger ones indefinitely (as long as the imposed shear provides the energy to sustain the process). Once the merging process takes over, the eddy structure remains fixed and exhibits highly directional integral scales (see 4 below).

3. In the model, the two transverse energy components do not separate while the data indicate that such a separation occurs. The splitting can, however, be brought into the model by assuming a tensor coefficient in the redistribution equation. We have taken this point to constitute a refinement at this stage of analysis.

4. The calculated angular averaged integral scales are quite directional. The model thus gives a picture of the eddy structure of a sheared turbulence. With our choice of parameters

$$\frac{\Lambda_{11}}{\Lambda_{22}} \sim 2.6 \quad , \quad \frac{\Lambda_{12}}{3\Lambda} \sim 0.19$$

this picture could eventually be tested by experiment.

APPENDIX A. SPECTRAL EQUATION

The equation for the spectral tensor ϕ_{ij} is

$$\frac{\partial\phi_{ij}}{\partial t} - U'K_1 \frac{\partial}{\partial K_2}\phi_{ij} =$$

$$= - U'(\phi_{i2}\delta_{j1} + \delta_{2i}\phi_{2j}) - i\left[\Phi_{ij}(\vec{K}) + \Phi_{ji}(-\vec{K})\right]$$

$$+ i\left[K_i P_j(\vec{K}) - K_j P_i(-\vec{K})\right] - 2\nu k^2 \phi_{ij} \tag{A.1}$$

where

$$\Phi_{ij}(\vec{K}) = K_\alpha \int S_{i\alpha j}(r)\, e^{-i\vec{K}\cdot\vec{r}} \frac{d\vec{r}}{(2\pi)^3}$$

$$P_j(\vec{K}) = \int \left\langle p'(x)u_j'(y)\right\rangle e^{-i\vec{K}\cdot\vec{r}} \frac{d\vec{r}}{(2\pi)^3}$$

$$S_{i\alpha j}(\vec{r}) = \left\langle u_i(x)u_\alpha(x)u_j(x + r)\right\rangle$$

Continuity requires the Poisson equation

$$ik^2 \frac{P_j(\vec{k}) - P_j(-\vec{K})}{2} = 2U'K_1\phi_{2j} + iK_\alpha\left[\Phi_{\alpha j}(\vec{K}) + \Phi_{\alpha j}(-\vec{K})\right] \tag{A.2}$$

The following simple model gives a tolerable picture for the energy spectrum of grid turbulence:

$$-i\left[\Phi_{ij}(\vec{k}) + \Phi_{ji}(-\vec{K})\right] =$$

$$= \frac{q}{\Lambda}\left[A\phi_{ij} + B \quad k\frac{\partial}{\partial k}\phi_{ij} + 2\phi_{ij} \quad + C\left(k^2\frac{\partial^2}{\partial k^2}\phi_{ij} + 4k\frac{\partial}{\partial k}\phi_{ij} + 2\phi_{ij}\right)\right]$$

$$\tag{A.3}$$

with

$$A \approx 0.075 \quad , \quad B \approx 0.615 \quad , \quad C \approx 0.27$$

which have been adjusted to give exactly a 5/3 law for equilibrium.

Even though a complete structure equation has not been determined for the sheared turbulence, it is seen that any closure scheme such as that given by Eq. (A.3) will have the exponential solution of the form given in Section III.

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